Normal subgroups:

Def. A subgroup $H \subset G$ is called normal if $aH = Ha$ for all $a \in G$. notation: $H \triangleleft G$

Examples:

① G abelian $\Rightarrow$ every subgroup $H \subset G$ is normal.

② If $[G:H]=2 \Rightarrow H \triangleleft G$
in particular $A_n \triangleleft S_n$

even permutations

③ $\langle (12) \rangle \subset S_3$ is NOT a normal subgroup.

Theorem (normal subgroup test)

Let $H \subset G$ be a subgroup

$H \trianglelefteq G$ normal $\iff$ $xHx^{-1} \subset H$ for all $x \in G$

proof. "$\Rightarrow$" Let $x \in G$, $h \in H$

H normal!

$$xh \in xH = Hx$$

$$\Rightarrow \exists\, h' \in H \text{ s.t. } xh = h'x \qquad \Big|\ \cdot x^{-1}$$

$$xhx^{-1} = h' \in H$$

$h \in H$ arbitrary $\Rightarrow xHx^{-1} \subset H$ ✓

"$\Leftarrow$"

given: $xHx^{-1} \subset H$ for all $x \in G$

let $x = a$: $aHa^{-1} \subset H \qquad \big|\ \cdot a$

$$\underbrace{aHa^{-1}a}_{= aH} \subset Ha$$

$$\Rightarrow aH \subset Ha$$

$\underline{x = a^{-1}:}$ $\quad a^{-1}Ha \subset H \quad | \, a\cdot$

$$aa^{-1}Ha \subset aH$$

$$\| $$

$$Ha$$

$\Rightarrow Ha \subset aH$

$\Rightarrow$ claim.

Example 4

Consider $H = \{id, (12)(34), (13)(24), (14)(23)\}$

Claim: H is a normal subgroup of A_4 and S_4

- H is a subgroup!

  do subgroup test.

  each element $\pi \in H$ satisfies $\pi^2 = id$

  $\Rightarrow \pi^{-1} = \pi \in H$ ✓

$\pi, \sigma \in H \overset{!}{\Rightarrow} \pi\sigma \in H$

done by brute force calculation.

e.g. $\underbrace{(12)(34)}_{\pi} \cdot \underbrace{(13)(24)}_{\sigma} = (14)(23) \in H$

check: H is an abelian subgroup
by doing all possible multiplications of elements of H

apply normal subgroup test.

to show: if $\pi \in H$, $\sigma \in S_4 \Rightarrow \sigma\pi\sigma^{-1} \in H$

If $\underbrace{(ij)(k,e)}_{\pi} \in H$ $\{i,j,k,e\} = \{1,2,3,4\}$

$\sigma(ij)(ke)\sigma^{-1} \underset{\text{Lemma}}{=} (\sigma(i)\,\sigma(j))\,(\sigma(k)\,\sigma(e))$

permutation with 2 disjoint 2-cycles

But H contains **all** permutations in S_4 of that form

$\Rightarrow \sigma(ij)(k\ell)\sigma^{-1} \in H$ for any $\sigma \in S_4$ and any $(ij)(k\ell) \in H$

$\Rightarrow H \triangleleft S_4$

and $H \triangleleft A_4$ (same proof)

(5) $G = GL(2,\mathbb{R})$

$H = SL(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}),\ \underbrace{\det(A) = 1}_{!}\}$

<u>claim:</u> $H \triangleleft G$

subgroup: $A, B \in SL(2,\mathbb{R})$

$AB \in SL(2,\mathbb{R})$?

need to calculate $\det(AB)$!

$$\det(AB) \underset{\uparrow \text{lin. algebra}}{=} \det(A)\det(B) \underset{\uparrow A,B \in SL(2,\mathbb{R})}{=} 1 \cdot 1 = 1$$

For inverse, we use $\det(A^{-1}) \underbrace{\det(A)}_{=1} = \det(A^{-1}A) = \det(I) = 1$

$\Rightarrow \det(A^{-1}) = 1$

$\Rightarrow A^{-1} \in SL(2,\mathbb{R})$

$\Rightarrow SL(2,\mathbb{R})$ is a subgroup.

normal?

to show: $B\,(SL(2,\mathbb{R}))\,B^{-1} \subset SL(2,\mathbb{R})$ for all $B \in GL(2,\mathbb{R})$

(by normal subgroup test)

Let $A \in SL(2,\mathbb{R})$ to show: $BAB^{-1} \in SL(2,\mathbb{R})$!

i.e. need to show $\det(BAB^{-1}) = 1$

$A \in SL(2,\mathbb{R})$

Lin. algebra $\Rightarrow$

$$\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(A)\det(B)^{-1} = \cancel{\det B}\,\cancel{(\det B)^{-1}}\,\det(A) = 1$$

Factor Groups

Theorem: Let $H \triangleleft G$

Then the set of cosets $\{aH, a \in G\}$ is a group G/H with operation $(aH)(bH) = abH$

example: $G = \mathbb{Z}$, $H = 2\mathbb{Z}$ even numbers

two cosets H and $1+H$

H ↑ even numbers

$1+H$ ↑ odd numbers

group op.

odd + odd = $(1+H)+(1+H) = (2+H) = H$ - even

odd + even = $(1+H) + H = 1+H$ = odd

...

Proof. important:

need to show that operation is well-defined!

i.e. if $\left.\begin{array}{l} aH = a'H \\ bH = b'H \end{array}\right\}$ for $a, a', b, b' \in G$ $\Rightarrow abH = a'b'H$

$aH = a'H \Rightarrow a' \in aH \Rightarrow a' = ah_1$

$bH = b'H \Rightarrow b' \in bH \Rightarrow b' = bh_2$

$h_1, h_2 \in H$

$$a'b'H = ah_1 b \underbrace{h_2 H}_{=H} = ah_1 bH = abh_3 H = abH$$

$\left(\begin{array}{l} h_3 H = \\ \{h_3 h, h \in H\} \\ = \{k, k \in H\} \\ = H \end{array} \right)$

H normal: $\Rightarrow Hb = bH$

$\Rightarrow \exists h_3 \in H$ s.t. $\boxed{\underset{\underset{Hb}{\uparrow}}{h_1 b} = bh_3}$